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# THRESHOLDS AND TRAVELLING WAVES FOR THE GEOGRAPHICAL SPREAD OF INFECTION

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Thresholds and travelling waves for the geographical spread of infection *)

by

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ABSTRACT

A nonlinear integral equation of mixed Volterra-Fredholm type describing the spatio-temporal development of an epidemic is derived and analysed. Particular attention is paid to the hair-trigger effect and to the travelling wave problem.

KEY WORDS & PHRASES: *spread of infection in space and time; nonlinear integral-equation of mixed Volterra-Fredholm type; nonlinear convolution equations; threshold phenomenon; hair-trigger effect; travelling waves.*

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*) This report will be submitted for publication elsewhere

## 1. INTRODUCTION

An important approach in mathematical population dynamics is to investigate which qualitative phenomena one can understand on the basis of some simple assumptions concerning an interaction mechanism. Kendall expressed this once as follows [12]: "I expect the usable results to be not numbers but a list of possible modes of behaviour". In this spirit we study in the present paper some qualitative aspects of the development of an epidemic in space and time.

In 1927, KERMACK and McKENDRICK [13] proposed a deterministic model for the evolution in time of a contageous disease in a closed population. The model leads to a nonlinear Volterra integral equation which can be analysed in great detail (see [13], [14] and [4]). Among other things, the analysis reveals that even in the long run a nontrivial fraction of the susceptible population escapes from getting the disease. This fraction is found from a root of a nonlinear scalar equation; some important conclusions can be drawn from the dependence of the relevant root on the various parameters.

A special case of the Kermack and McKendrick model leads to a system of ordinary differential equations. KENDALL [11] has introduced a space-dependent analogue of this system by taking for the infectivity at each point a weighted spatial average of the density of infectives. In this case, the final fraction of susceptibles is a function of position; it is found from a solution of a nonlinear integral equation. On the basis of this equation Kendall made it plausible that, for certain values of the parameters, the ultimate effect of the introduction of an arbitrary small amount of infectivity at some point would be large at every point of the spatial domain. The system exhibits therefore what ARONSON and WEINBERGER [2] call the hair-trigger effect. In this paper we consider the space-dependent analogue of the general Kermack and McKendrick model and we prove the hair-trigger effect for one - and two-dimensional spatial domains.

A topic that has received considerable attention is the travelling wave problem. As far as we know, all work done so far takes the Kendall system as a starting point. MOLLISON [15] proves the existence of travelling waves for a special weight function which enables him to reduce the problem to a system of ordinary differential equations and then to use invariant manifold theory. KENDALL [12] arrives at a system of ordinary differential equations by replacing the spatial average by a diffusion approximation (see also [8] and [10]). It seems that ATKINSON and REUTER [3] were the first to prove existence and nonexistence of travelling waves by analyzing the integro-differential system itself. In this paper we deal with the travelling wave problem for the general Kermack and McKendrick model. This problem leads us to an integral equation which can be treated in the style of Atkinson and Reuter.

As a general introduction to the field of mathematical epidemiology we refer to the lecture notes of HOPPENSTEADT [10]. As far as mathematics is concerned, the main subject of this paper is the study of nonlinear convolution equations on the real line. This work was begun in [4] and will be continued in [5].

## 2. THE MODEL

Consider a population living in a habitat $\Omega$ (a closed subset of $\mathbb{R}^n$) and susceptible to some contageous disease. Our first aim is to give a heuristic derivation of an equation for the evolution of an epidemic among the population on the basis of some simple assumptions. The only dynamical phenomenon that we are going to consider is infection. So, according to the custom to study but one interaction mechanism at a time, we simply ignore the effect of changes due to birth, migration, etcetera. We assume that space dependence manifests itself through the possibility that an infectious individual at position $x_1$ can infect a susceptible individual at position $x_2$. In this connection one may think of such phenomena as forest-fires, the wind borne dissemination of spores that cause plant-diseases, or contagion among animals (or people) walking around by day and returning to their home for the night.

Let $S(t,x)$ and $I(t,x)$ denote the density of *susceptibles* and *infectives*, respectively, at time $t$ and position $x$. Let $i(t,\tau,x)d\tau$ be the density of infectives which were infected some time between $t-\tau$ and $t-\tau-d\tau$. Then,

$$(2.1) \qquad I(t,x) = \int_0^\infty i(t,\tau,x)d\tau.$$

We suppose that the population size is large, so we can consider the variables S, I and i as continuous, or even continuously differentiable, real-valued functions of their arguments.

Let the *infectivity* $B = B(t,x)$ be defined as the rate at which susceptibles become infective. The basic assumptions in this paper are that

(i) the disease induces permanent immunity, so the transition from I to S does not occur,

(ii) a nonnegative function $A = A(\tau,x,\xi)$ is given such that

$$B(t,x) = \int_0^\infty \int_\Omega i(t,\tau,\xi)A(\tau,x,\xi)\,d\xi\,d\tau. \tag{2.2}$$

Thus, $A(\tau,x,\xi)$ describes the infectivity at x due to one infective of "age of illness" $\tau$ at $\xi$. Many characteristics of both the disease and the habitat are in fact incorporated in A.

The above definitions and assumptions lead to the following system of dynamical equations:

$$\frac{\partial S}{\partial t}(t,x) = -S(t,x)B(t,x), \tag{2.3}$$

$$\frac{\partial S}{\partial t}(t,x) = -i(t,0,x), \tag{2.4}$$

$$i(t,\tau,x) = i(t-\tau,0,x). \tag{2.5}$$

Elimination of i from (2.2) by means of (2.4) and (2.5), followed by substitution of the result into (2.3) yields an equation for S alone,

$$\frac{\partial S}{\partial t}(t,x) = S(t,x)\int_0^\infty \int_\Omega \frac{\partial S}{\partial t}(t-\tau,\xi)A(\tau,x,\xi)\,d\xi\,d\tau. \tag{2.6}$$

A solution S of (2.6) has to be defined at least for $-\infty < t \le T$ for some finite T. In section 6, equation (2.6) will be taken as the starting point for an analysis of travelling wave solutions.

An appropriate *initial value problem* is obtained by prescribing

$$i(0,\tau,x) = i_0(\tau,x) \text{ and } S(0,x) = S_0(x)$$

and by restricting the validity of the dynamical equations to $t > 0$. Then, instead of (2.6), we obtain

$$\frac{\partial S}{\partial t}(t,x) = S(t,x)\left\{\int_0^t\int_\Omega \frac{\partial S}{\partial t}(t-\tau,\xi)A(\tau,x,\xi)\,d\xi\,d\tau - h(t,x)\right\}, \tag{2.7}$$

where

$$h(t,x) = \int_0^\infty\int_\Omega i_0(\tau,\xi)A(t+\tau,x,\xi)\,d\xi\,d\tau. \tag{2.8}$$

Finally, assuming $S_0(x) > 0$ for every $x \in \Omega$, we integrate equation (2.7) once with respect to t. If we interchange the order of the integrations, we can write the resulting equation in the form

$$u(t,x) = \int_0^t\int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)\,d\xi\,d\tau + f(t,x), \tag{2.9}$$

where

$$u(t,x) = -\ln\frac{S(t,x)}{S_0(x)}, \tag{2.10}$$

$$g(y) = 1 - e^{-y}, \tag{2.11}$$

$$f(t,x) = \int_0^t h(\tau,x)\,d\tau. \tag{2.12}$$

In the next section we shall discuss the problems of existence, uniqueness and asymptotic behaviour of solutions of equation (2.9). To some extent the specific form of the forcing function f will be irrelevant; all we shall need is the fact that f is a given nonnegative function which is monotonic nondecreasing with respect to t.

To conclude this section we make some remarks concerning the biological aspects of the model and we discuss some special cases.

REMARK 2.1. If $\Omega$ is bounded, the influence of the boundary has to be incorporated in the function A. How this is done depends on the underlying mechanism. For instance, if $\Omega$ is a cornfield and the wind is acting as a driving force for the spread of spores, then the boundary $\partial\Omega$ has no active influence on the spread and all spores falling outside $\Omega$ are simply without any effect. However, if $\Omega$ is a town, then individuals living near the boundary are attracted by the centre and, consequently, A will include anisotropic boundary effects. Likewise, internal inhomogeneities of $\Omega$ (mountains, rivers) can give rise to anisotropy in A.

REMARK 2.2. Another feature of A is its coupling of the age of illness $\tau$ and the space variables x and $\xi$. Again, this coupling varies with the type of the underlying mechanism. An important special case arises if we assume that there is no coupling at all,

$$A(\tau,x,\xi) = H(\tau)W(x,\xi) \quad , \tag{2.13}$$

where we can normalize so that

$$\int_\Omega W(x,\xi)dx = 1 \quad . \tag{2.14}$$

The situation can be simplified even further if we assume that the function W depends on the distance only,

$$W(x,\xi) = V(x-\xi) \quad , \tag{2.15}$$

where V is a radial function.

REMARK 2.3. Two special cases of equation (2.7) have received much attention in the literature. Both cases have in common that $\Omega = \mathbb{R}^n$, that (2.13) - (2.15) hold, and that $i_0(\tau,x) = j_0(x)\delta(\tau)$ (introduction of fresh infectives only; $\delta$ is the Dirac delta function), with $S_0(x) + j_0(x) = 1$ (constant population density). With these assumptions (2.7) can be written as

$$\frac{\partial S}{\partial t}(t,x) = S(t,x)\int_{\mathbb{R}^n}\left\{\int_0^t \frac{\partial S}{\partial t}(t-\tau,\xi)H(\tau)d\tau - H(t)j_0(\xi)\right\}V(x-\xi)d\xi .$$

Two special cases are:

a) *a simple epidemic* ([3], [15], [16]): $H(t) \equiv a$.
Then

$$\frac{\partial S}{\partial t}(t,x) = aS(t,x)\{\bar{S}(t,x) - 1\} \quad ,$$

where $\bar{S}$ is a weighted spatial average of $S$,

$$\bar{S}(t,x) = \int_{\mathbb{R}^n} S(t,\xi)V(x-\xi)d\xi.$$

b) *a general epidemic* ([3], [8], [10], [11], [12]): $H(t) = ae^{-\mu t}$.
Define

$$Y(t,x) = -\int_0^t \frac{\partial S}{\partial t}(\tau,x)e^{-\mu(t-\tau)}d\tau + j_0(x)e^{-\mu t}.$$

Then $Y$ and $S$ satisfy the system of equations

$$\frac{\partial Y}{\partial t}(t,x) = -\frac{\partial S}{\partial t}(t,x) - \mu Y(t,x) \quad ,$$

$$\frac{\partial S}{\partial t}(t,x) = -aS(t,x)\bar{Y}(t,x) \quad ,$$

where $\bar{Y}$ is the weighted spatial average of $Y$, as defined above. This system admits another interpretation: $Y$ is the density of the infectives, every infective has constant infectivity $a$, and the infectives are removed with rate $\mu$.

## 3. THE INITIAL VALUE PROBLEM

The nonlinear integral equation

$$(3.1) \qquad u(t,x) = \int_0^t \int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)d\xi d\tau + f(t,x),$$

is of Volterra type with respect to t and of Fredholm type with respect to x. In making the assumptions concerning A, g, $S_0$ and f we shall be guided by the biological interpretation and by the desire to let the analysis proceed along standard lines. We do not aim at generality.

Let $BC(\Omega)$ denote the Banach space of the bounded continuous functions on $\Omega$ equipped with the supremum norm. Then a convenient framework for the study of (3.1) is provided by the Banach space $C_T = C([0,T];BC(\Omega))$ of continuous functions on $[0,T]$ with values in $BC(\Omega)$, equipped with the norm

$$\|f\|_{C_T} = \sup_{0\le t\le T} \|f[t]\|_{BC(\Omega)} .$$

When looking at u as an element of $C_T$ we shall write $u[t](x)$ instead of $u(t,x)$. With this convention we can write (3.1) as

$$u[t] = Qu[t] + f[t] \quad , \tag{3.2}$$

where Q is defined by

$$Qu[t](x) = \int_0^t \int_\Omega g(u[t-\tau](\xi))S_0(\xi)A(\tau,x,\xi)d\xi d\tau . \tag{3.3}$$

As will be proved in Lemma 3.1 below, the following assumptions guarantee that Q is a mapping from $C_T$ into $C_T$.

$A_{S_0}$ : $S_0 \in L_\infty(\Omega)$ ; $S_0$ is nonnegative.

$A_g$ : $g:\mathbb{R} \to \mathbb{R}$ is continuous ; $g(0) = 0$.

$A_A^1$ : $A(\cdot,\cdot,\cdot)$ is defined and nonnegative on $[0,\infty)\times\Omega\times\Omega$ ; for every $x \in \Omega$ and every $T > 0$, $A(\cdot,x,\cdot) \in L_1([0,T]\times\Omega)$.

$A_A^2$ : Let $\eta(t,x) = \int_0^t \int_\Omega A(\tau,x,\xi)d\xi d\tau$ and let $T > 0$ be arbitrary, then the family of functions on $[0,T]$, $\{\eta(\cdot,x) \mid x \in \Omega\}$ is uniformly bounded and equicontinuous.

$A_A^3$ : For every $T > 0$ and every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,T) > 0$ such that if $x_1, x_2 \in \Omega$ and $|x_1 - x_2| < \delta$ then

$$\int_0^T \int_\Omega |A(\tau,x_1,\xi) - A(\tau,x_2,\xi)| d\xi d\tau < \varepsilon .$$

For notational convenience we define

$$\bar{S}_0 = \sup_{x\in\Omega} S_0(x) \quad .$$

LEMMA 3.1. *For every* $T > 0$ *and every* $u \in C_T$, $Qu \in C_T$.

PROOF. Let $T > 0$ and $u \in C_T$ be arbitrary and put $k = \|u\|_{C_T}$. First we prove that for fixed $t \in [0,T]$, $Qu[t] \in BC(\Omega)$. The boundedness follows from

$$|Qu[t](x)| \le \sup_{|y|\le k} |g(y)| \cdot \bar{S}_0 \cdot \int_0^T \int_\Omega A(\tau,x,\xi) d\xi d\tau \quad ,$$

and the uniform boundedness assumption in $A_A^2$. The continuity with respect to x follows from

$$|Qu[t](x_1) - Qu[t](x_2)| \le \sup_{|y|\le k} |g(y)| \cdot \bar{S}_0 \cdot \int_0^T \int_\Omega |A(\tau,x_1,\xi) - A(\tau,x_2,\xi)| d\xi d\tau$$

and $A_A^3$.

Next we show that the mapping $t \mapsto Qu[t]$ from $\mathbb{R}$ to $BC(\Omega)$ is continuous. Suppose $0 \le t_2 \le t_1 \le T$, then

$$\sup_{x\in\Omega} | Qu[t_1](x) - Qu[t_2](x) | \le$$

$$\bar{S}_0 \sup_{x\in\Omega} \int_0^T \int_\Omega A(\tau,x,\xi) d\xi d\tau \cdot \sup_{\substack{\xi\in\Omega \\ \tau\in[0,t_2]}} |g(u[t_1 - t_2 + \tau](\xi)) - g(u[\tau](\xi))| +$$

$$\sup_{|y|\le k} |g(y)| \cdot \bar{S}_0 \cdot \sup_{x\in\Omega} \int_{t_2}^{t_1} \int_\Omega A(\tau,x,\xi) d\xi d\tau .$$

If $t_1 - t_2 \downarrow 0$ then both terms tend to zero. For the first term this follows from $A_A^2$ and the uniform continuity of $t \mapsto u[t]$ on $[0,T]$ and $y \mapsto g(y)$ on $|y| \le k$, and for the second term this follows from the equicontinuity assumption in $A_A^2$. □

So if we assume that f satisfies

$A_f$: $\qquad f: [0,\infty) \to BC(\Omega)$ is continuous,

then equation (3.2) makes sense in $C_T$ for arbitrary $T > 0$.

<u>THEOREM</u> 3.2. *(local existence and uniqueness).*
*Suppose* g *is locally Lipschitz continuous, then there exists a* $T > 0$ *such that (3.2) has a unique solution* u *in* $C_T$ *and the mapping* $f \mapsto u$ *is continuous from* $C_T$ *into* $C_T$.

<u>PROOF</u>. Consider the closed subset of $C_T$

$$X_{f,T} = \{h \in C_T \mid \|h - f\|_{C_T} \le 1\}$$

and let the mapping $R_f: C_T \to C_T$ be defined by

$$R_f u = Qu + f.$$

Choose $T_0 > 0$ and let $\ell$ be the Lipschitz constant of g corresponding to the set $|y| \le 1 + \|f\|_{C_{T_0}}$. Since $\|f\|_{C_T}$ is a nondecreasing function of T we have for all $T < T_0$ and $u \in X_{f,T}$,

$$\|u\|_{C_T} \le 1 + \|f\|_{C_{T_0}} .$$

Let $u,v \in X_{f,T}$ then

$$\|Qu\|_{C_T} \le \ell \bar{S}_0 \sup_{x\in\Omega} \int_0^T \int_\Omega A(\tau,x,\xi)d\xi d\tau . \ (1 + \|f\|_{C_{T_0}}),$$

and

$$\|R_f u - R_f v\|_{C_T} \le \ell \bar{S}_0 \sup_{x\in\Omega} \int_0^T \int_\Omega A(\tau,x,\xi)d\xi d\tau . \|u-v\|_{C_T} .$$

So if

$$\sup_{x\in\Omega} \int_0^T \int_\Omega A(\tau,x,\xi)d\xi d\tau < (\ell \bar{S}_0(1 + \|f\|_{C_{T_0}}))^{-1} ,$$

then $R_f$ is a contraction mapping of $X_{f,T}$ into itself. From $A_A^2$ we conclude that we can choose indeed $T \in (0,T_0]$ such that the inequality holds and then the first half of the theorem follows from the Banach contraction mapping theorem. This existence proof is constructive: if $u_0 = f$ and $u_{n+1} = Qu_n + f$, then $u_n$ converges as $n \to \infty$ in $C_T$ to the solution u.

In order to prove the continuity of the mapping $f \mapsto u$ one needs to modify the foregoing proof just a little to show that for a given $f \in C_{T_0}$ there exist a neighbourhood Y of f in $C_{T_0}$, a $T \in (0,T_0]$ and a closed subset $X_{f,T}$ such that $R_{\tilde{f}}$ is a contraction mapping of $X_{f,T}$ into $X_{f,T}$ uniformly for $\tilde{f} \in Y$ (we leave the details of this modification to the reader). Clearly the mapping $f \mapsto R_f u$ is continuous on $C_T$ for each $u \in C_T$ and, consequently, the continuous dependence of the fixed point on f follows from [9, Theorem 0.3.2]. We remark that one can also prove the continuous dependence by making use of Gronwall's inequality [9, Corollary I.6.6] and a slightly different assumption concerning A. □

Once we know a solution of (3.2) in $C_{T_1}$ we can write

$$\tilde{u}[t] = Q\tilde{u}[t] + \tilde{f}[t] ,$$

with

$$\tilde{u}[t] = u[T_1 + t] ,$$

$$\tilde{f}[t](x) = f[T_1 + t](x) + \int_0^{T_1} \int_\Omega g(u[\tau](\xi))S_0(\xi)A(T_1 + t-\tau,x,\xi)d\xi d\tau .$$

Application of Theorem 3.2 then yields a continuation of u in $C_{T_2}$, $T_2 > T_1$, and by repetition of the argument one finds a maximally defined solution (compare [9, section I.2]).

THEOREM 3.3. *(global existence and uniqueness).*
*Suppose* g *is uniformly Lipschitz continuous then (3.2) has a unique continuous solution* $u: [0,\infty) \to BC(\Omega)$.

PROOF. In the continuation procedure one can choose $T_{n+1} - T_n$ independent of n. □

We now turn our attention to some further properties of the solution that arise as a consequence of the structure of the forcing function f and the properties of g in the biological model. On $BC(\Omega)$ we introduce a partial ordering by

$$\phi \geq \psi \text{ if and only if } \phi(x) \geq \psi(x) \text{ for all } x \in \Omega.$$

THEOREM 3.4. *(positivity and monotonicity).*

*(a)* *Suppose* $g(y) > 0$ *for* $y > 0$ *and* $f[t] \geq 0$ *for all* $t \geq 0$, *then* $u[t] \geq 0$ *on the domain of definition of* u.

*(b)* *Suppose, in addition,* g *is monotone nondecreasing and* $f[t+h] \geq f[t]$ *for all* $h \geq 0$, *then* $u[t+h] \geq u[t]$ *for all* $h \geq 0$ *and* $t \geq 0$ *such that* $t + h$ *is in the domain of definition of* u.

PROOF. Both (a) and (b) follow from the construction of the solution (see the proof of Theorem 3.2). □

If f is given by (2.12) and g by (2.11) then they have the properties that are assumed in Theorem 3.4. The conclusion is biologically evident, since we assumed that the disease induces permanent immunity.

Next we discuss the boundedness and the asymptotic behaviour as $t \to \infty$, of a globally defined solution. If g, f and $\eta$ (see $A_A^2$) are bounded from above, then a simple estimate shows that u is bounded from above. In combination with Theorem 3.4(a), (b) this yields the convergence of $u[t](x)$ to a finite limit as $t \to \infty$, pointwise in x. By imposing further conditions on A and f we can strengthen the convergence and we can deduce an equation for the limit.

THEOREM 3.5. *(stabilization).*
*In addition to the assumptions of Theorem 3.4(a), (b), suppose that* g *is bounded and uniformly Lipschitz continuous on* $[0,\infty)$, *that the subset*

$\{f[t] \mid t \geq 0\}$ *of* $BC(\Omega)$ *is uniformly bounded and equicontinuous and that* A *satisfies*

$A_A^4$: *For each* $x \in \Omega$

$$\int_0^t A(\tau,x,\cdot)d\tau \to \int_0^\infty A(\tau,x,\cdot)d\tau \text{ in } L_1(\Omega) \text{ as } t \to \infty,$$

*and for some* $C > 0$

$$\sup_{x\in\Omega} \int_\Omega \int_0^\infty A(\tau,x,\xi)d\tau d\xi < C.$$

$A_A^5$: *For each* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that if* $x_1,x_2 \in \Omega$ *and* $|x_1 - x_2| < \delta$ *then*

$$\int_\Omega \int_0^\infty |A(\tau,x_1,\xi) - A(\tau,x_2,\xi)| \, d\tau d\xi < \varepsilon.$$

*Then the solution* u *of (3.2) is defined on* $[0,\infty)$ *and there exists* $u[\infty] \in BC(\Omega)$ *such that, as* $t \to \infty$, $u[t] \to u[\infty]$ *in* $BC(\Omega)$ *if* $\Omega$ *is compact, and uniformly on compact subsets of* $\Omega$ *if* $\Omega$ *is not compact. Moreover* $u[\infty]$ *satisfies the limit equation*

$$u[\infty](x) = \int_\Omega g(u[\infty](\xi))S_0(\xi) \int_0^\infty A(\tau,x,\xi)d\tau d\xi + f[\infty](x). \tag{3.4}$$

PROOF. The global existence follows from Theorem 3.4(a) and the uniform Lipschitz continuity of g on $[0,\infty)$. We have noticed before that $u[t](x)$ converges pointwise in x. The estimate

$$|u[t](x_1) - u[t](x_2)| \leq \sup_{y\geq 0} g(y)\cdot \bar{S}_0 \cdot \int_\Omega \int_0^\infty |A(\tau,x_1,\xi) - A(\tau,x_2,\xi)| d\tau d\xi + \\ + |f[t](x_1) - f[t](x_2)| \quad ,$$

shows that the subset $\{u[t] \mid t \geq 0\}$ of $BC(\Omega)$ is equicontinuous. Hence the Arzela-Ascoli theorem can be applied to strengthen the convergence as stated in the theorem. We leave it to the reader to write down the straightforward but rather tedious estimates that show that one may take limits in the equation in order to get (3.4). □

In biological terms $u[\infty]$ is the fraction of the susceptible population that escapes from getting the disease. Therefore we call $u[\infty]$ the *final size* and (3.4) the *final size equation*.

The assumptions conserning A are rather technical. However, in the special case that (2.13) - (2.15) hold they simplify considerably: if $\Omega = \mathbb{R}^n$, $V \in L_1(\mathbb{R}^n)$ and $H \in L_1([0,T])$ for all $T > 0$, then $A_A^1 - A_A^3$ are satisfied (recall that translation is continuous in $L_1$), and if $H \in L_1([0,\infty))$ then $A_A^4$ and $A_A^5$ are satisfied as well.

## 4. THE FINAL SIZE EQUATION

The analysis in [4] of the initial value problem for the space-independent model proceeds along more or less the same lines as followed in section 3. There the final size equation is a scalar equation

(4.1) $$x(\infty) = \gamma s_0 g(x(\infty)) + f(\infty) \quad ,$$

which can be analysed, for instance graphically. With g given by (2.11) one finds that (4.1) has a unique positive solution $x(\infty)$ for each positive $f(\infty)$. The dependence of $x(\infty)$ on the parameters $s_0$ and $\gamma$ is best illustrated by looking at

$$\underline{x} = \inf_{f(\infty)>0} x(\infty).$$

It follows that $\underline{x}$ satisfies the homogeneous equation

(4.2) $$y = \gamma s_0 g(y)$$

and $\underline{x}$ is positive if and only if $\gamma s_0 > 1$. This is the so-called *threshold phenomenon* which has received considerable attention (see [13], [4]).

The investigation of the final size equation (3.4) for the space-dependent model is not so simple. However, it is still possible to show that there is an analogous threshold phenomenon. The analysis requires one extra step: with (3.4) we shall associate a scalar equation (which is essentially the same as in the space-independent case) and then we shall show that a positive solution of the scalar equation serves pointwise as a lower bound for any bounded, nonnegative solution of (3.4).

Put

$$s_0 = \inf_{x\in\Omega} S_0(x)$$

and

$$\gamma = \inf_{x\in\Omega} \int_\Omega \int_0^\infty A(\tau,x,\xi)\,d\tau\,d\xi$$

Thus, $\gamma$ is the infimum of the total infectivity at x due to a homogeneously distributed unit density of infectives during the whole course of the disease. The scalar equation we have in mind is (4.2). With

$$\underline{u} = \inf_{x\in\Omega} u[\infty](x) \quad ,$$

we obtain from (3.4) and the monotonicity of g

$$(4.3) \qquad \underline{u} \geq s_0\gamma g(\underline{u})$$

and this makes it tempting to conjecture that $\underline{u} \geq p$, where p is defined as the positive solution of (4.2) if $\gamma s_0 > 1$ and $p = 0$ if $\gamma s_0 \leq 1$. This is sometimes called Kendall's Pandemic Threshold Theorem (see [11]).

<u>THEOREM</u> 4.1. *Assume that*

*(i)* $\Omega$ *is compact and connected,*

*(ii)* $s_0\gamma g(y) > y$ *for* $0 < y < p$ ,

*(iii) fcr each* $x \in \Omega$ *there exists* $\delta = \delta(x)$ *such that the set* $\{\xi \mid |x-\xi| \leq \delta\} \cap \Omega$ *is contained in the support of*

$$\int_0^\infty A(\tau,x,\cdot)\,d\tau.$$

*If* $f[\infty] \in BC(\Omega)$ *is nonnegative, but not identically zero, then every non-negative solution* $u[\infty] \in BC(\Omega)$ *of (3.4) satisfies* $u[\infty] \geq p$.

PROOF. From (ii) and (4.3) we obtain that either $\underline{u} = 0$ or $\underline{u} \geq p$. Suppose $\underline{u} = 0$. Since $u[\infty]$ achieves its infimum, there exists $x_0 \in \Omega$ such that $u[\infty](x_0) = 0$. In (3.4) both terms are nonnegative, so (iii) implies that $u[\infty](x) = 0$ for $x \in \{\xi \mid |x_0 - \xi| \leq \delta(x_0)\} \cap \Omega$. Repeating the same argument one finds a maximal subset X of $\Omega$ where $u[\infty](x) = 0$. Necessarily $\partial X \subset \partial\Omega$ and hence $X = \Omega$. But with $u[\infty] \equiv 0$, (3.4) cannot be satisfied, and our assumption $\underline{u} = 0$ must be false. □

So the conjecture is seen to be true for compact $\Omega$. The conditions, that $\Omega$ is connected and that A satisfies (iii) of Theorem 4.1, express that $\Omega$ should not consist of parts which are isolated with respect to infection.

Mathematically, the problem is much more interesting if $\Omega$ is not compact. Restricting our attention to the special case $\Omega = \mathbb{R}^n$, and assuming that (2.13) - (2.15) hold, we obtain the following inequality from (3.4)

$$u[\infty](x) \geq s_0\gamma \int_{\mathbb{R}^n} g(u[\infty](\xi))V(x-\xi)d\xi.$$

This leads to the investigation of the bounded solutions of the equation

$$w(x) = \int_{\mathbb{R}^n} g(w(\xi))V(x-\xi)d\xi + h(x),$$

with $h(x) \geq 0$. (We have incorporated the constant $s_0\gamma$ in the function g.) The results in the next section will imply the correctness of the conjecture for $n = 1$ or $n = 2$.

It is worthwhile to keep in mind that the lower bound p is independent of the function $f[t]$ as long as f is not identically zero. Hence, if $s_0\gamma > 1$, then equation (3.1) manifests the hair-trigger effect: no matter how little infectivity is introduced in an arbitrarily small subset of $\Omega$, eventually there will be a large effect at every point.

## 5. A NONLINEAR CONVOLUTION EQUATION

Consider the nonlinear convolution equation

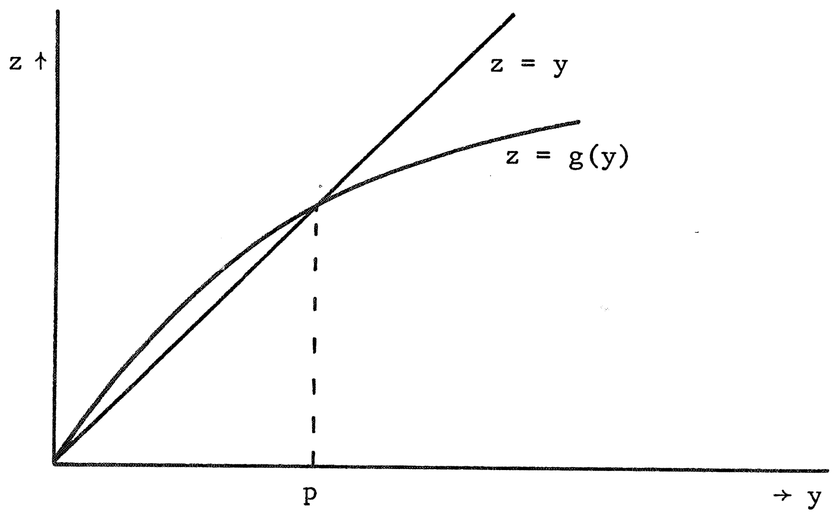
(5.1) $$w(x) = \int_{\mathbb{R}^n} g(w(\xi))V(x-\xi)d\xi + h(x) ,$$

for the unknown function $w: \mathbb{R}^n \to \mathbb{R}$, where $n = 1$ or $n = 2$. The functions g, V and h satisfy the following hypotheses:

$H_g$: $g: \mathbb{R} \to \mathbb{R}$ is continuous and monotone nondecreasing; $g(0) = 0$ and the equation

(5.2) $$y = g(y)$$

has one positive root p. Moreover, $g(y) > y$ for $0 < y < p$ and $g(y) < y$ for $y > p$.




$H_V$: $V \in L_1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} V(x)dx = 1$ ; $\int_{\mathbb{R}^n} |x|^n V(x)dx < \infty$ ; ($|\cdot|$ denotes the Euclidean norm)

V is a nonnegative radial function.

$H_h$: $h: \mathbb{R}^n \to \mathbb{R}$ is continuous, nonnegative and not identically zero.

Under only slightly more restrictive assumptions the existence of a bounded nonnegative solution of (5.1) can be established by means of a monotone iteration process. But we have a different aim in view, viz. to prove that p is a lower bound for any such solution.

THEOREM 5.1. *Let* $w\colon \mathbb{R}^n \to \mathbb{R}$ *be a continuous nonnegative solution of (5.1). Then* $w(x) \geq p$ *for every* $x \in \mathbb{R}^n$.

PROOF. Let v be defined by $v(x) = \min\{w(x),p\}$. Then (5.1) implies

$$v(x) \geq \int_{\mathbb{R}^n} v(\xi)V(x-\xi)d\xi. \tag{5.3}$$

It is known that in fact the inequality has to be an equality. A proof of this result can be found in ESSÉN [6, Theorem 3.1] or in FELLER [7, sections VI.10 and XVIII.7]. The basic idea of that proof is to show that the existence of a bounded function v satisfying (5.3) with strict inequality in a set of positive measure would imply the convergence of the infinite sum

$$\sum_{k=1}^{\infty} \int_{|x|\leq h} V^{k*}(x)dx$$

for some $h > 0$ (here $V^{k*}$ denotes the $(k-1)$-times iterated convolution of V with itself), whereas the vanishing of the expectation(s) of V (recall that V is a radial function) implies the divergence of the same infinite sum for each $h > 0$. The latter result is proved by means of Fourier analysis. Note that here the dimension n and the assumptions concerning the existence of moments of V are important.

The only bounded continuous solutions of the equation

$$v(x) = \int_{\mathbb{R}^n} v(\xi)V(x-\xi)d\xi \quad ,$$

are constants (see [6, Theorem 2.1] or [7, section XI.2]). Thus, $v(x) \equiv c$. The case $c = 0$ can be excluded, because then $w(x) \equiv 0$ and, hence, (5.1) cannot be satisfied. If $0 < c < p$, then the inequality (5.3) is strict. So $c = p$ and the theorem is proved. □

The following result will be needed in section 6.

THEOREM 5.2. *Let* $w\colon \mathbb{R}^n \to \mathbb{R}$ *be a bounded nonnegative solution of the homogeneous equation*

$$(5.4) \qquad w(x) = \int_{\mathbb{R}^n} g(w(\xi))V(x-\xi)d\xi \ .$$

*Then either* $w(x) \equiv 0$ *or* $w(x) \equiv p$.

PROOF. As in the proof of Theorem 5.1 it follows that either $w(x) \equiv 0$ or $w(x) \geq p$ for every $x \in \mathbb{R}^n$. Let

$$\beta = \sup_{x\in\mathbb{R}^n} w(x) \quad ,$$

then (5.4) implies $w(x) \leq g(\beta)$. Suppose $\beta > p$, then

$$w(x) \leq g(\beta) < \beta + \frac{1}{2}(g(\beta) - \beta).$$

Hence $\beta - w(x)$ is bounded away from zero which is in contradiction with the definition of $\beta$. □

One can also consider the convolution equation (5.1) with a Borel probability measure instead of a $L_1$-kernel. The results of this section remain true if it is assumed that the support of the measure is non-arithmetic (or non-lattice, i.e., the support should not lie in any cyclic subgroup of $\mathbb{R}^n$; see [6], [7]).

The hypothesis concerning g was chosen to cover the case $g(y) = s_0\gamma(1-\exp(-y))$ and to avoid any multiplicity complication. If, on the other hand, the graph of $z = g(y)$ winds itself around the straight line $z = y$, then the homogeneous equation (5.4) may have many more bounded solutions. However, the nonexistence of nontrivial solutions satisfying certain bounds can still be shown along the same lines.

Thus far we have assumed that the population density exceeds the threshold on all of $\mathbb{R}^n$. Suppose, on the contrary, that the threshold is exceeded only on a half-space, then the question arises whether the introduction of infectives on the other half-space will give rise to a severe epidemic. The following result answers this question positively.

THEOREM 5.3. *Let* $w: \mathbb{R}^n \to \mathbb{R}$ *be a bounded nonnegative solution of*

$$(5.5) \qquad w(x) = \int_{\mathbb{R}^n} g(w(\xi))q(\xi)V(x-\xi)d\xi + h(x).$$

*In addition to* $H_g$, $H_V$ *and* $H_h$ *assume that:*

*(i)* *The support of* $V$ *contains an interval if* $n = 1$ *and a ring or a circle if* $n = 2$,

*(ii)* *there exist* $\delta > 0$, $\ell > 1$ *such that* $g(x) \geq \ell x$ *for* $0 \leq x \leq \delta$,

*(iii)* $q \in L_\infty(\mathbb{R}^n)$ *and* $q$ *is strictly positive.*

*Then*

*(a)* *if* $n = 1$, $\liminf_{x \to \infty} q(x) > 1$ *implies* $\liminf_{x \to \infty} w(x) \geq p$.

*(b)* *if* $n = 2$, *the existence of a sector* $\Lambda$ *such that* $q(x) > 1$ *asymptotically for* $|x| \to \infty$ *in* $\Lambda$ *implies that* $w(x) \geq p$ *asymptotically for* $|x| \to \infty$ *in every sector* $\Lambda'$ *that is contained in* $\Lambda$.

PROOF. Since the arguments are exactly the same in both cases, we shall only give the proof of the result in case $n = 1$. We split the proof into several steps. Some of the arguments in step 6 are inspired by the discussion of Kendall [11].

STEP 1. The minimal nonnegative solution of (5.5) arises by monotone iteration with $h(x)$ as a starting function. Therefore it is sufficient to prove the result for a function $h$ which has a compact support (for the minimal solution for arbitrary $h$ is pointwise greater than or equal to the minimal solution corresponding to a compact supported $\tilde{h}$ which satisfies $\tilde{h} \leq h$). So let us assume that $h$ has compact support.

STEP 2. $u$ is uniformly continuous. For the first term on the right-hand side of (5.5) the uniform continuity follows from the continuity of translation in $L_1$ and for the second term from our assumption that $h$ has compact support.

STEP 3. Let the limit set $Q$ be defined by

$Q = \{\zeta \mid$ there exists a sequence $\{x_n\}$ with $x_n \to \infty$ and $w(x_n) \to \zeta$ as $n \to \infty\}$.

If $\zeta \in Q$, then either $\zeta = 0$ or $\zeta \geq p$.
This can be seen as follows. Suppose $w(x_n) \to \zeta$ and $x_n \to \infty$ as $n \to \infty$. By the Arzela-Ascoli theorem, the sequence of translated functions $w(x+x_n)$ has a subsequence which converges uniformly on compact subsets to a bounded, non-negative function $v_\zeta(x)$ say. Let $\bar{x}$ be such that $q(x) \geq 1$ for $x \geq \bar{x}$. From

$$w(x+x_n) \geq \int_{-\infty}^{x+x_n-\bar{x}} g(w(x+x_n-\xi))V(\xi)d\xi$$

we obtain upon passing to the limit

$$v_\zeta(x) \geq \int_{-\infty}^{\infty} g(v_\zeta(x-\xi))V(\xi)d\xi.$$

Hence either $v_\zeta(x) \equiv 0$ and $\zeta = 0$ or $v_\zeta(x) \geq p$ for every $x \in \mathbb{R}^n$ and $\zeta \geq p$ (Theorem 5.1).

STEP 4. Q is connected (intermediate value argument). So if $0 \in Q$ then $Q = \{0\}$ or, in other words, $\lim_{x\to\infty} w(x) = 0$.

STEP 5. w is strictly positive.
Indeed, suppose $w(x_0) = 0$ for some $x_0$, then from (5.5) it follows that $w(x) = 0$ for all x such that $x - x_0$ is contained in the support of V and, by assumption, this set contains an interval. By repetition of the argument we can cover the whole line and hence $w(x_0) = 0$ implies $w(x) \equiv 0$, which is impossible in view of (5.5).

STEP 6. It is impossible that $\lim_{x\to\infty} w(x) = 0$.
Suppose $\lim_{x\to\infty} w(x) = 0$. Then there exists $\tilde{x} = \tilde{x}(\delta)$ such that $w(x) \leq \delta$ for all $x \geq \tilde{x}$ and consequently $g(w(x)) \geq \ell w(x)$ for $x \geq \tilde{x}$. For every $d > 0$ (5.5) implies

$$w(x) \geq \int_{-d}^{d} g(w(x-\xi))V(\xi)d\xi, \quad \text{if } x \geq \tilde{x} + d,$$

so if $x \geq a+d$, where $a = \max\{\tilde{x},\bar{x}\}$, then

$$w(x) \geq \ell \int_{-d}^{d} w(x-\xi)V(\xi)d\xi \ ,$$

and by iteration, if $x \geq a+md$, then

$$(5.6) \qquad w(x) \geq \ell^m \int_{-md}^{md} w(x-\xi)V_d^{m*}(\xi)d\xi.$$

Here $V_d^{m*}$ is the $(m-1)$-times iterated convolution of the truncated kernel V with itself (the symmetry of the kernel is conserved under convolution with itself!).

Since w is positive we can define for $n > (w(0))^{-1}$,

$$x_n = \sup\{x \mid w(\xi) \geq n^{-1} \text{ for } 0 \leq \xi \leq x\}.$$

It follows that $w(x_n) = n^{-1}$, $w(x) \geq w(x_n)$ for $0 \leq x \leq x_n$ and furthermore $x_n \to \infty$ for $n \to \infty$. At this moment we make a particular choice for the parameters d, m and n:

d so large that $\theta > 1$, where $\theta = \ell \int_{-d}^{d} V(\xi)d\xi$,

m so large that $\theta^m > 2$,

n so large that $x_n \geq a+md$.

Then we obtain from (5.6)

$$w(x_n) \geq \ell^m \int_{0}^{md} w(x_n-\xi)V_d^{m*}(\xi)d\xi \geq \frac{1}{2}\,\theta^m\, w(x_n) > w(x_n),$$

which is impossible.

Finally, combination of the results stated in the steps 3, 4 and 6 yields that if $\zeta \in Q$ then $\zeta \geq p$ and consequently $\liminf_{x\to\infty} w(x) \geq p$. □

In conclusion of this section we remark that one can use the same kind of arguments as have been used in the proof of Theorem 5.3 in the analysis of the final size equation for non-compact $\Omega$ other than $\mathbb{R}^n$, for instance for half-lines or half-planes.

## 6. TRAVELLING WAVES

Now that we have shown that solutions of the initial value problem stabilize, we may ask how they approach the final distribution. Instead of studying this question in general, we concentrate on a special case and a particular set of solutions.

Let $\Omega = \mathbb{R}$ and $A(\tau,x,\xi) = \gamma H(\tau)V(x-\xi)$. Throughout this section we shall assume that

(i) $H \in L_1([0,\infty))$; $\int_0^\infty H(\tau)d\tau = 1$; $H$ is nonnegative;

(ii) $V \in L_1(\mathbb{R})$ ; $\int_{-\infty}^\infty V(x)dx = 1$; $V$ is nonnegative; $V(-x) = V(x)$;

if $W_\lambda(x) = e^{-\lambda x}V(x)$, then $W_\lambda \in L_1(\mathbb{R})$ for $\lambda$ in a vertical strip (of positive width) of the complex plane.

We take the time-translation invariant equation (2.6) as our starting point. Let $S(-\infty,x) = S$ be a given constant (so we assume constant population density). We integrate equation (2.6) once with respect to t. Thus we obtain an equation of the form

$$(6.1) \qquad u(t,x) = \int_0^\infty H(\tau)\int_{-\infty}^\infty g(u(t-\tau,\xi))V(x-\xi)d\xi d\tau$$

for

$$u(t,x) = -\ell n\,\frac{S(t,x)}{S} ,$$

where

$$(6.2) \qquad g(y) = \gamma S(1-e^{-y}).$$

A *travelling wave solution* of (6.1) is a solution of the form $u(t,x) = w(x+ct)$. One may visualize a travelling wave solution as a function of x which, as time increases, is propagated (to the left if $c > 0$ and to the right if $c < 0$) with constant velocity c without any alteration in shape.

Since a travelling wave w depends on a linear combination of the independent variables x and t, one expects that it is possible to deduce an equation for w in one independent variable. Indeed, this is easily achieved by substitution and rearranging:

$$(6.3) \qquad w(\xi) = \int_{-\infty}^{\infty} g(w(\eta))V_c(\xi - \eta)d\eta, \quad \xi = x + ct,$$

where

$$(6.4) \qquad V_c(\xi) = \int_0^{\infty} H(\tau)V(\xi - c\tau)d\tau, \quad -\infty < \xi < \infty.$$

Incidentally we observe that a simlar equation arises in the study of plane wave solutions in higher dimensions.

For every c, (6.3) is a homogeneous nonlinear convolution equation with a nonnegative kernel. The constant solutions of (6.3) are the roots of

$$(6.5) \qquad y = g(y)$$

and we assume that (6.5) has, in addition to the root $y = 0$, one positive root $y = p$ (in view of (6.2) this amounts to the assumption $\gamma S > 1$).

If $c = 0$ (equilibrium solutions) the kernel $V_c$ is symmetric and we may apply Theorem 5.2 to conclude that $w \equiv 0$ and $w \equiv p$ are the only bounded nonnegative solutions of (6.3). The question arises whether or not there are values of c for which (6.3) has a solution satisfying $0 < w(\xi) < p$ and how these values of c can be characterized. From the symmetry of V it is immediately clear that, if $w(\xi)$ satisfies (6.3) for $c = \bar{c}$, then $w(-\xi)$ satisfies (6.3) for $c = -\bar{c}$, so we can confine ourselves to the case $c > 0$. In order to get a first impression we note that

$$\int_{-\infty}^{\infty} V_c(\xi)d\xi = 1$$

and

$$\int_{-\infty}^{\infty} \xi V_c(\xi)d\xi = c \int_0^{\infty} \tau H(\tau)d\tau,$$

if the integral at the right-hand side converges. Or, in words: the total mass of $V_c$ is constant, but the distribution of mass becomes more and more lopsided as c increases. It turns out that (6.3) may have a nontrivial solution if $V_c$ is lopsided enough.

Consider the linearized (about $w \equiv 0$) equation

$$v(\xi) = g'(0) \int_{-\infty}^{\infty} v(\eta)V_c(\xi - \eta)d\eta \tag{6.6}$$

and the corresponding characteristic equation

$$L_c(\lambda) = 1, \tag{6.7}$$

where

$$L_c(\lambda) = g'(0) \int_{-\infty}^{\infty} e^{-\lambda\xi}V_c(\xi)d\xi = g'(0) \int_0^{\infty} e^{-\lambda c\tau}H(\tau)d\tau . \int_{-\infty}^{\infty} e^{-\lambda\xi}V(\xi)d\xi. \tag{6.8}$$

If $\lambda_0$ is a root of equation (6.7) of multiplicity $k \geq 1$, then $\xi^m \exp(\lambda_0\xi)$ is a solution of (6.6) for $m = 0,1,\ldots,k-1$ (one can verify this by substitution) and essentially all solutions of (6.6) are generated in this manner (see TITCHMARSH [17, Theorem 146] and [4]). The observation that positive solutions of (6.6) correspond to real roots of (6.7) now motivates us to study $L_c$ as a function of a real variable and to list some of its properties:

(i) $L_c(\lambda)$ is defined in a right-hand neighbourhood of $\lambda = 0$;

(ii) $L_c(0) = g'(0) > 1$ (by assumption);

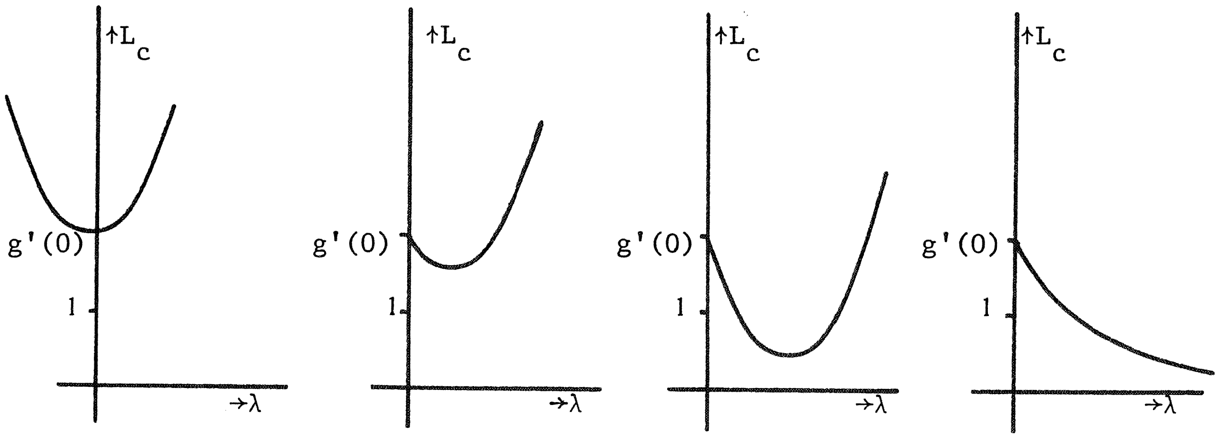
(iii) $\frac{dL_c}{d\lambda}(0+) = -cg'(0) \int_0^{\infty} \tau H(\tau)d\tau < 0$ (possibly $-\infty$);

(iv) $\frac{d^2L_c}{d\lambda^2}(\lambda) = g'(0) \int_{-\infty}^{\infty} \xi^2 e^{-\lambda\xi}V_c(\xi)d\xi > 0$, i.e., $L_c$ is a convex function.

Next we describe the dependence on c. If $c = 0$, $L_c(\lambda)$ has a minimum for $\lambda = 0$. From (iii) and (iv) it follows that for $c > 0$ a minimum can occur only for positive $\lambda$. For every $\lambda > 0$, $L_c(\lambda)$ is a monotonically decreasing function of c, and we can achieve that $L_c(\lambda) < 1$ by choosing c sufficiently large. Consequently, the set

$$\{c \mid \text{there exists } \lambda > 0 \text{ such that } L_c(\lambda) < 1\}$$

consists of a half-line, $(c_0,\infty)$ say.




$c = 0$ $\qquad$ $0 < c < c_0$ $\qquad$ $c > c_0$ $\quad$ or $\quad$ $c > c_0$

We have provided enough motivation to state the following theorem.

THEOREM 6.1. *Let* $g\colon \mathbb{R} \to \mathbb{R}$ *be continuous and monotone nondecreasing and suppose* $g(0) = 0$, $g(y) > y$ *for* $0 < y < p$ *and* $g(p) = p$. *Assume, furthermore, that* $g'(0)$ *exists and that, for some* $k > 0$,

$$g'(0)y - ky^2 \le g(y) \le g'(0)y, \qquad \text{for } 0 \le y \le p.$$

*Let* $K \in L_1(\mathbb{R})$ *be nonnegative and let* $\int_{-\infty}^{\infty} K(x)dx = 1$. *Suppose there exists* $\lambda_0 > 0$ *for which* $L(\lambda_0) < 1$, *where*

$$L(\lambda) = g'(0) \int_{-\infty}^{\infty} e^{-\lambda x} K(x)dx.$$

*(Note that the convergence of the integral for* $\lambda = \lambda_0$ *is part of the assumption.)*
*Then the equation*

(6.9) $$w(x) = \int_{-\infty}^{\infty} g(w(\eta))K(x-\eta)d\eta$$

*has a monotone nondecreasing solution* w *which has the following properties,*

$$0 < w(x) < p, \lim_{x\to-\infty} w(x) = 0, \lim_{x\to+\infty} w(x) = p.$$

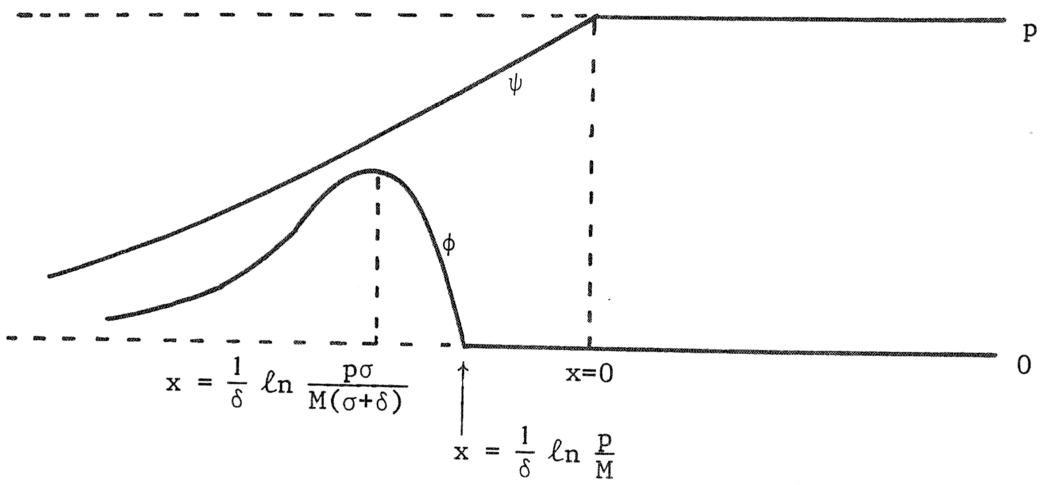
REMARK. The subjoined proof is essentially the same as the proof of a related result in [3], but it is different in a number of details.

PROOF. The convergence of the defining integral for $L(\lambda)$ for $\lambda = 0$ and $\lambda = \lambda_0$ implies the convergence for $0 < \lambda < \lambda_0$. Since $L(\lambda)$ is continuous and $L(0) > 1$, there exists a $\sigma \in (0,\lambda_0)$ such that $L(\sigma) = 1$ and, since L is convex, $L(\lambda) < 1$ for $\sigma < \lambda \leq \lambda_0$. We introduce the following functions

$$\psi(x) = \min\{pe^{\sigma x},p\} \quad ,$$

$$\phi(x) = \max\{pe^{\sigma x}-Me^{(\sigma+\delta)x},0\} \quad ,$$

with $M,\delta > 0$ still at our disposal.



A first requirement we want to meet is that $\phi(x) < \psi(x)$ for all $x \in \mathbb{R}$. If the maximum of $p\exp(\sigma x) - M\exp(\sigma x + \delta x)$ is achieved for negative x then this will certainly be the case and thus we find as a sufficient condition

$$(6.10) \qquad M > \frac{\sigma p}{\sigma + \delta} \quad .$$

Defining a formal integral-operator T by the right-hand side of (6.9), we now try to choose M and $\delta$ so that $T\psi \le \psi$ and $T\phi \ge \phi$. The former inequality follows directly from the assumptions and from the equality $L(\sigma) = 1$, so we concentrate on the latter.

The properties of g imply the estimate

$$(T\phi)(x) \ \ge \ g'(0) \int_{-\infty}^{\infty} \phi(x-\eta)K(\eta)d\eta \ - \ k \int_{-\infty}^{\infty} \phi^2(x-\eta)K(\eta)d\eta .$$

By choosing $0 < \delta \le \sigma$ we can achieve that

$$\phi^2(x) \ \le \ \psi^2(x) \ \le \ p^2 \ e^{(\sigma+\delta)x} \quad ,$$

and hence

$$(T\phi)(x) \ \ge \ p \ e^{\sigma x} \ - \ ML(\sigma+\delta)e^{(\sigma+\delta)x} \ - \ \frac{kp^2}{g'(0)} L(\sigma+\delta)e^{(\sigma+\delta)x} \quad .$$

Consequently,

$$(T\phi)(x) \ \ge \ p \ e^{\sigma x} \ - \ Me^{(\sigma+\delta)x} \quad ,$$

provided

$$(6.11) \qquad ML(\sigma+\delta) \ + \ \frac{kp^2}{g'(0)} L(\sigma+\delta) \ \le M \quad .$$

Choose $\delta$ such that $0 < \delta \le \min\{\sigma, \lambda_0 - \sigma\}$, then $L(\sigma+\delta) < 1$ and we can satisfy (6.10) and (6.11) by choosing M sufficiently large that

$$M \ \ge \max\left\{ \frac{kp^2}{g'(0)} \ (1 - L(\sigma+\delta))^{-1}, \ \frac{\sigma p}{\sigma+\delta} \right\} \quad .$$

Then indeed $T\phi \geq \phi$ (note that $T\phi \geq 0$ is trivial).

The operator T is monotone, i.e., $\chi_1 \leq \chi_2$ implies $T\chi_1 \leq T\chi_2$. Starting with $\psi$ and iterating with T we obtain a monotone decreasing sequence $\{\psi_n = T^n\psi \mid n = 0,1,\dots\}$ which is bounded from below by $\phi$ and, hence, pointwise convergent. As before, the Arzela-Ascoli theorem can be applied to show that the convergence is in fact uniform on compact subsets, which implies that the limit, say w, satisfies the equation (6.9). Since $\psi$ is monotone non-decreasing, the same is true for $T^n\psi$ and, hence, for w. From the properties of $\psi$ and $\phi$ we obtain $\lim_{x\to-\infty} w(x) = 0$. The monotonicity and the boundedness of w imply the existence of $\lim_{x\to+\infty} w(x)$ and, finally, a translation argument shows that this limit is a constant solution of (6.9) and, thus, equal to p. □

<u>COROLLARY</u> 6.2. *Let* g *be given by (6.2) with* $\gamma S > 1$. *Then there exists a* $c_0 > 0$ *such that (6.1) has a travelling wave solution for every* $c > c_0$. *The constant* $c_0$ *is given by* $c_0 = \inf\{c \mid \text{there exists } \lambda > 0 \text{ such that } L_c(\lambda) < 1\}$.

In [5], the nonlinear convolution equation (6.9) is studied and Tauberian methods are used to prove a nonexistence result. Under some reasonable conditions on H and V this yields a complementary nonexistence result for travelling waves with $0 < c < c_0$. The question whether or not there is a travelling wave solution for $c = c_0$ appears to be more delicate.

The results of [5] suggest that, under some additional conditions, travelling waves for fixed c are unique modulo translation. A special case deserves attention: if the support of V is compact and contained in $[-a,a]$, and the support of H is contained in $[b,\infty)$ with $b > 0$, then for $c \geq ab^{-1}$ the support of $V_c$ is contained in $[0,\infty)$ and the uniqueness of the travelling wave follows from [4, section 4].

An obvious continuation of the above analysis would be to study the stability of the travelling waves, that is, their relation to solutions of the initial value problem (3.1). A related approach, with a somewhat different flavour, would be to investigate the relation between $c_0$ and the asymptotic speed of propagation of infectivity starting from a compactly supported initial function. Here, ARONSON and WEINBERGER [1], [2] could be a source of inspiration. Some work on the rate of propagation of infectivity has already been done by MOLLISON [16].

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